A PROOF OF THE PROPORTIONALITY THEOREM

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The Proportionality Theorem of [BS] establishes a deep relation between the Schwartz index of stratified vector fields on Whitney stratified complex analytic varieties and the liftings of these vector fields to sections of the Nash bundle. This is one of the key ingredients for the proof in [BS] that the Alexander isomorphism carries the Schwartz classes of singular varieties to the corresponding MacPherson classes.

Recently we considered the corresponding problem for 1-forms and gave a proof of the Proportionality Theorem in [BSS]. It is inspired by the original proof for vector fields in [BS], however is shorter and more direct. Then, coming back to the case of vector fields, we realized that the original proof, for vector fields and for frames as well, can be substantially simplified.

The purpose of this note is to give a direct and self-contained proof of the Proportionality Theorem in order to facilitate the understanding of this important theorem.

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1. The case of vector fields

Let V be a complex analytic variety of pure dimension n in a complex manifold M of dimension m. We endow M with a Whitney stratification $\{V_\alpha\}$ adapted to V. Let SV denote the union of the tangent bundles of all the strata; $SV = \bigcup_\alpha TV_\alpha$, which is thought of as a subset of the tangent bundle TM of M. A *stratified vector field* is a section of TM whose image is in SV. Here TM denotes

the complex tangent bundle of M, however, recall that it is canonically isomorphic to the real tangent bundle as a real bundle.

The Nash modification and the Nash bundle are constructed as follows. Let $\text{Sing}(V)$ denote the singular set and $V_{\text{reg}} = V \setminus \text{Sing}(V)$ the regular part of V. We have a map $\sigma : V_{\text{reg}} \to G_n(TM)$ into the Grassmann bundle of complex n planes in TM, which assigns to each point p in V_{reg} the subspace $T_p V_{\text{reg}}$ of $T_p M$. The Nash modification $\widetilde{V}$ is the closure of the image of σ together with the restriction $\nu : \widetilde{V} \to V$ of the projection $G_n(TM) \to M$ to $\widetilde{V}$. The Nash bundle $\widetilde{T} \to \widetilde{V}$ is the restriction to $\widetilde{V}$ of the tautological bundle over $G_n(TM)$. Note that $\widetilde{T}$ is a subbundle of $\nu^* TM|_V$ and is isomorphic to TV_{reg} away from $\nu^{-1}(\text{Sing}(V))$.

If v is a non-vanishing stratified vector field on a subset A in V, then by the Whitney condition (a), it can be lifted to a non-vanishing section $\widetilde{v}$ of $\widetilde{T}$ over $\nu^{-1}A$ (see [BS]). Namely, the pull-back $\nu^* v$, which is a priori a section of $\nu^* TM|_A$, is in fact a section of $\widetilde{T}$. We denote it by $\widetilde{v}$ to emphasize that it is a section of $\widetilde{T}$.

We take a closed ball $\mathbb{B}$ in M around a point p in V and let $\mathbb{S} = \partial \mathbb{B}$. If v is a stratified vector field, which is non-vanishing on $\mathbb{S} \cap V$, it can be lifted to a non-vanishing section $\widetilde{v}$ of $\widetilde{T}$ over $\nu^{-1}(\mathbb{S} \cap V)$, as noted above. Let $o(\widetilde{v})$ denote the class in $H^{2n}(\nu^{-1}(\mathbb{B} \cap V), \nu^{-1}(\mathbb{S} \cap V))$ of the obstruction cocycle to extending this to a non-vanishing section over $\nu^{-1}(\mathbb{B} \cap V)$.

Definition 1.1. The *local Euler obstruction* $\text{Eu}(v, V; p)$ of a stratified vector field v at an isolated singularity p is the integer obtained by evaluating $o(\widetilde{v})$ on the orientation cycle $[\nu^{-1}(\mathbb{B} \cap V), \nu^{-1}(\mathbb{S} \cap V)]$, for sufficiently small $\mathbb{B}$.

As a particular vector field satisfying the previous situation, one can consider a radial vector field, v_{rad}, i.e. a vector field pointing outwards $\mathbb{B}$ along $\mathbb{S}$. By [BS], the local Euler obstruction $\text{Eu}(v_{\text{rad}}, V; p)$ for a stratified radial vector field coincides with the *local Euler obstruction of V at p*, $\text{Eu}(V, p)$, introduced by MacPherson [M]; we take this as our definition of $\text{Eu}(V, p)$.

Now let v_α be a vector field on a stratum V_α with an isolated singularity at $p \in V_\alpha$. Recall that by the radial extension process of M.-H. Schwartz [Sc], v_α can be extended to a vector field v'_α defined in a neighborhood of p in M. In the rest of this section, we denote the vector field v'_α simply by v. The vector field v is stratified with an isolated singularity at p. Moreover, it has the property that its Poincaré-Hopf index $\text{Ind}_{\text{PH}}(v, M; p)$ in M coincides with the Poincaré-Hopf index $\text{Ind}_{\text{PH}}(v_\alpha, V_\alpha; p)$ of v_α in V_α, provided that $\dim V_\alpha > 0$. If $\dim V_\alpha = 0$, then v is a radial vector field in the usual sense, and we have $\text{Ind}_{\text{PH}}(v, M; p) = 1$.

Definition 1.2. Let v be as above. The *Schwartz index* $\text{Ind}_{\text{Sch}}(v, V; p)$ of v relative to V at p is the Poincaré-Hopf index $\text{Ind}_{\text{PH}}(v, M; p)$ of v as a vector field on M, or equivalently, if $\dim V_\alpha > 0$, the Poincaré-Hopf index $\text{Ind}_{\text{PH}}(v_\alpha, V_\alpha; p)$ of v_α on V_α.

First we give the proposed proof of the Proportionality Theorem in [BS] in the case of vector fields.

Theorem 1.3. *Let $V_\alpha \subset V$ be a stratum and v_α a vector field on V_α with an isolated singularity at p. Let v denote the radial extension of v_α. Then we have*

$$\mathrm{Eu}(v, V; p) = \mathrm{Eu}(V, p) \cdot \mathrm{Ind}_{\mathrm{Sch}}(v, V; p).$$

Proof. In the sequel, we denote by $T^\times M$ and $\widetilde{T}^\times$ the spaces obtained from TM and $\widetilde{T}$ by removing the zero sections.

First we recall (Definition 1.2) that

$$(1.4) \qquad\qquad \mathrm{Ind}_{\mathrm{Sch}}(v, V; p) = \mathrm{Ind}_{\mathrm{PH}}(v, M; p).$$

Let v_{rad} denote a stratified radial vector field at p. Then, by definition of $\mathrm{Ind}_{\mathrm{PH}}(v, M; p)$, there is a homotopy

$$\Psi : \mathbb{S} \times [0, 1] \longrightarrow T^\times M|_{\mathbb{S}}$$

such that

$$\partial \, \mathrm{Im}\, \Psi = v(\mathbb{S}) - \mathrm{Ind}_{\mathrm{PH}}(v, M; p) \cdot v_{\mathrm{rad}}(\mathbb{S})$$

as chains in $T^\times M|_{\mathbb{S}}$. Moreover, since the vector fields v and v_{rad} are both stratified and the stratification is Whitney, we can assume that the above homotopy Ψ is through stratified vector fields, i.e., we may choose Ψ so that

$$(1.5) \qquad\qquad \mathrm{Im}\, \Psi \subset SV.$$

The restriction of Ψ gives a homotopy

$$\psi : (\mathbb{S} \cap V) \times [0, 1] \longrightarrow T^\times M|_{\mathbb{S} \cap V}$$

such that (cf. (1.4))

$$\partial \, \mathrm{Im}\, \psi = v(\mathbb{S} \cap V) - \mathrm{Ind}_{\mathrm{Sch}}(v, V; p) \cdot v_{\mathrm{rad}}(\mathbb{S} \cap V).$$

We can lift v and v_{rad} to sections $\nu^* v$ and $\nu^* v_{\mathrm{rad}}$ of $\nu^* T^\times M|_{\nu^{-1}(\mathbb{S} \cap V)}$; we can also lift ψ to a homotopy

$$\nu^* \psi : \nu^{-1}(\mathbb{S} \cap V) \times [0, 1] \longrightarrow \nu^* T^\times M|_{\nu^{-1}(\mathbb{S} \cap V)}$$

and we have

$$\partial \, \mathrm{Im}\, \nu^* \psi = \nu^* v(\nu^{-1}(\mathbb{S} \cap V)) - \mathrm{Ind}_{\mathrm{Sch}}(v, V; p) \cdot \nu^* v_{\mathrm{rad}}(\nu^{-1}(\mathbb{S} \cap V))$$

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as chains in $\nu^* T^\times M|_{\nu^{-1}(\mathbb{S} \cap V)}$. By (1.5), everything can be restricted to $\widetilde{T} \subset \nu^* TM$ to get a homotopy

$$\widetilde{\psi} : \nu^{-1}(\mathbb{S} \cap V) \times [0, 1] \to \widetilde{T}^\times|_{\nu^{-1}(\mathbb{S} \cap V)}$$

and we have

$$\partial \operatorname{Im} \widetilde{\psi} = \widetilde{v}(\nu^{-1}(\mathbb{S} \cap V)) - \operatorname{Ind}_{\mathrm{Sch}}(v, V; p) \cdot \widetilde{v}_{\mathrm{rad}}(\nu^{-1}(\mathbb{S} \cap V))$$

as chains in $\widetilde{T}^\times|_{\nu^{-1}(\mathbb{S} \cap V)}$.

Taking a triangulation or a cellular decomposition of $\nu^{-1}(\mathbb{B} \cap V)$ and extending the homotopy $\widetilde{\psi}$ to the $(2n-1)$-skeleton of the decomposition, we see that the obstruction to extending $\widetilde{v}$ is $\operatorname{Ind}_{\mathrm{Sch}}(v, V; p)$ times the obstruction to extending $\widetilde{v}_{\mathrm{rad}}$. By definition of the Euler obstructions, we have the theorem. $\square$

2. The case of frames

Let M, V and $\{V_\alpha\}$ be as in Section 1. We take a triangulation (K) of M compatible with the stratification and let (D) denote the cellular decomposition dual to (K). Note that the cells in (D) are transverse to V and V_α so that if σ denotes a cell of real dimension $2s$, then $\sigma \cap V$ and $\sigma \cap V_a$ are of dimensions $2(s - m + n)$ and $2(s - m + n_\alpha)$, respectively, where $n_\alpha = \dim_{\mathbb{C}} V_\alpha$.

In the sequel, an r-field means a collection $v^{(r)} = (v_1, \ldots, v_r)$ of r vector fields. A singular point of $v^{(r)}$ is a point where the vectors fail to be linearly independent (over the complex numbers). An r-frame is an r-field without singularity.

Let $\nu : \widetilde{V} \to V$ be the Nash modification of V. Let σ be a cell of dimension $2(m - r + 1)$ and $v^{(r)}$ a stratified r-field on $\sigma \cap V$ with an isolated singularity at the barycenter p of σ. Since $v^{(r)}$ is non-singular on $\partial\sigma \cap V$, it can be lifted to an r-frame $\widetilde{v}^{(r)}$ of $\widetilde{T}$ over $\nu^{-1}(\partial\sigma \cap V)$, as in the case of vector fields. Let $o(\widetilde{v}^{(r)})$ denote the class in $H^{2(n-r+1)}(\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V))$ of the obstruction cocycle to extending this to an r-frame over $\nu^{-1}(\sigma \cap V)$.

Definition 2.1. The *local Euler obstruction* $\operatorname{Eu}(v^{(r)}, V; p)$ of a stratified r-field $v^{(r)}$ at an isolated singularity p is the integer obtained by evaluationg $o(\widetilde{v}^{(r)})$ on the orientation cycle $[\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V)]$.

Now let $v_\alpha^{(r)}$ be an r-field on $\sigma \cap V_\alpha$ with an isolated singularity at the barycenter p of σ, where σ is a cell of dimension $2(m - r + 1)$. We may construct an r-field $v_\alpha^{(r)'}$ on σ by the radial extension process of Schwartz [Sc]. In the sequel, we denote $v_\alpha^{(r)'}$ simply by $v^{(r)}$. The r-field $v^{(r)}$ is stratified with an isolated singularity at p. Moreover, it has the property that its index $\operatorname{Ind}(v^{(r)}, M; p)$ (the obstruction to

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extending $v^{(r)}$ as an r-frame of TM on σ) coincides with the index $\mathrm{Ind}(v_\alpha^{(r)}, V_\alpha; p)$ (the obstruction to extending $v_\alpha^{(r)}$ as an r-frame of TV_α on $\sigma \cap V_\alpha$), provided that $\dim_{\mathbb{C}} V_\alpha \geq r$. If $\dim V_\alpha = r - 1$, then $v^{(r)}$ is a radial r-field in the usual sense, and we have $\mathrm{Ind}_{\mathrm{PH}}(v^{(r)}, M; p) = 1$.

Definition 2.2. Let $v^{(r)}$ and σ be as above. The *Schwartz index* $\mathrm{Ind}_{\mathrm{Sch}}(v^{(r)}, V; p)$ of $v^{(r)}$ relative to V at an isolated singularity p is the index $\mathrm{Ind}(v^{(r)}, M; p)$ of $v^{(r)}$ as an r-field on σ, or equivalently, the index $\mathrm{Ind}(v_\alpha^{(r)}, V_\alpha; p)$ of $v_\alpha^{(r)}$ on $\sigma \cap V_\alpha$, if $\dim V_\alpha \geq r$.

We now give the proposed proof of the Proportionality Theorem in [BS] in the case of frames.

Theorem 2.3. *Let $V_\alpha \subset V$ be a stratum and $v_\alpha^{(r)}$ an r-field on $\sigma \cap V_\alpha$ with an isolated singularity at the barycenter p of σ, where σ is a cell of dimension $2(m - r + 1)$. Let $v^{(r)}$ denote an r-field on σ obtained from $v_\alpha^{(r)}$ by radial extension. Then we have*

$$\mathrm{Eu}(v^{(r)}, V; p) = \mathrm{Eu}(V, p) \cdot \mathrm{Ind}_{\mathrm{Sch}}(v^{(r)}, V; p).$$

Proof. If $r = 1$, this is Theorem 1.3. If $r > 1$, we reduce the problem to the case $r = 1$ in the following way.

First, we may write $v^{(r)}$ as $(v^{(r-1)}, v_r)$, where the $(r-1)$-field $v^{(r-1)}$ is non-singular on σ. Let E denote the trivial subbundle of $TM|_\sigma$ of rank $r - 1$ spanned by $v^{(r-1)}$ (over the complex numbers) and Q the orthogonal complement of E in $TM|_\sigma$ for some metric;

$$TM|_\sigma = E \oplus Q.$$

Accordingly, we have a decomposition on $\nu^{-1}\sigma$:

$$\nu^* TM|_\sigma = \nu^* E \oplus \nu^* Q.$$

Since the r-field $(v^{(r-1)}, v_r)$ is stratified and non-singular on $\partial\sigma \cap V$, it lifts to an r-frame $\widetilde{v}^{(r)} = (\widetilde{v}^{(r-1)}, \widetilde{v}_r)$ of the Nash bundle $\widetilde{T}$ over $\nu^{-1}(\partial\sigma \cap V)$. Moreover, since $v^{(r-1)}$ is non-singular on $\sigma \cap V$, $\nu^* E$ (restricted to $\nu^{-1}(\sigma \cap V)$) is a subbundle of $\widetilde{T}|_{\nu^{-1}(\sigma \cap V)}$ and we have a decomposition:

$$(2.4) \qquad\qquad \widetilde{T}|_{\nu^{-1}(\sigma \cap V)} = \nu^* E \oplus \widetilde{P},$$

where $\widetilde{P}$ is a subbundle of $\nu^* Q|_{\nu^{-1}(\sigma \cap V)}$. We may think of $\widetilde{v}_r$ as a section of $\widetilde{P}$ which is non-vanishing on $\nu^{-1}(\partial\sigma \cap V)$. If we denote by $o(\widetilde{v}_r, \widetilde{P})$ the class in

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$H^{2(n-r+1)}(\nu^{-1}(\sigma\cap V),\nu^{-1}(\partial\sigma\cap V))$ of the obstruction cocycle to extending this to a non-vanishing section over $\nu^{-1}(\sigma\cap V)$, we have

$$(2.5) \qquad \mathrm{Eu}(v^{(r)},V;p)=o(\widetilde{v}_r,\widetilde{P})[\nu^{-1}(\sigma\cap V),\nu^{-1}(\partial\sigma\cap V)].$$

Since the vector field v_r is defined only on σ, we complement it by a radial vector field on the complementary space, in order to apply Theorem 1.3. Thus let $\mathbb{D}$ denote a small closed disc of complex dimension $r-1$ with center p and transverse to σ and set $\mathbb{B}=\mathbb{D}\times\sigma$. The bundle E extends to a neighborhood of p in M and we get, denoting also by E and Q the extensions of E and Q, a decomposition

$$TM|_{\mathbb{B}}=E\oplus Q.$$

The bundles E and Q can be interpreted as $\pi_1^*T\mathbb{D}$ and $\pi_2^*T\sigma$, respectively, where $\pi_1:\mathbb{B}\to\mathbb{D}$ and $\pi_2:\mathbb{B}\to\sigma$ denote the projections. Since E is in SV, i.e., the vectors in E are stratified, the pull-back ν^*E (restricted to $\nu^{-1}(\mathbb{B}\cap V)$) is a subbundle of $\widetilde{T}|_{\nu^{-1}(\mathbb{B}\cap V)}$ and we have a decomposition

$$\widetilde{T}|_{\nu^{-1}(\mathbb{B}\cap V)}=\nu^*E\oplus\widetilde{P},$$

where $\widetilde{P}$ is a subbundle of $\nu^*Q|_{\nu^{-1}(\mathbb{B}\cap V)}$, extending $\widetilde{P}$ in (2.4).

Now we may think of v_r as a section of $T\sigma$. Let $v_{\mathbb{D}}$ denote a radial vector field on $\mathbb{D}$ at p. Then the direct sum $v_0=(\pi_1^*v_{\mathbb{D}},\pi_2^*v_r)$ is a stratified vector field at p. Let $\widetilde{v}_0=(\widetilde{\pi_1^*v_{\mathbb{D}}},\widetilde{\pi_2^*v_r})$ be the lifting of these vector fields to sections of $\widetilde{T}$ over $\nu^{-1}(\mathbb{S}\cap V)$, where $\widetilde{\pi_1^*v_{\mathbb{D}}}=\nu^*\pi_1^*v_{\mathbb{D}}$ is a section of ν^*E and $\widetilde{\pi_2^*v_r}$ is a section of $\widetilde{P}$. If we denote by $o(\widetilde{v}_0,\widetilde{T})$ the class of obstruction cocycle to extending $\widetilde{v}_0$ to a non-vanishing section over $\nu^{-1}(\mathbb{B}\cap V)$, by definition we have

$$\mathrm{Eu}(v_0,V;p)=o(\widetilde{v}_0,\widetilde{T})[\nu^{-1}(\mathbb{B}\cap V),\nu^{-1}(\mathbb{S}\cap V)].$$

We now show the identity

$$(2.6) \qquad \mathrm{Eu}(v^{(r)},V;p)=\mathrm{Eu}(v_0,V;p).$$

Denoting by $o(\nu^*\pi_1^*v_{\mathbb{D}},\nu^*E)$ the class of obstruction cocycle to extending $\nu^*\pi_1^*v_{\mathbb{D}}$, a section of ν^*E non-vanishing on $\nu^{-1}((\partial\mathbb{D}\times\sigma)\cap V)=\nu^{-1}(\partial\mathbb{D}\times(\sigma\cap V))$, to a non-vanishing section on $\nu^{-1}(\mathbb{B}\cap V)$ and by $o(\widetilde{\pi_2^*v_r},\widetilde{P})$ the class of obstruction cocycle to extending $\widetilde{\pi_2^*v_r}$, a section of $\widetilde{P}$ non-vanishing on $\nu^{-1}((\mathbb{D}\times\partial\sigma)\cap V)=\nu^{-1}(\mathbb{D}\times(\partial\sigma\cap V))$, to a non-vanishing section on $\nu^{-1}(\mathbb{B}\cap V)$, we have

$$o(\widetilde{v}_0,\widetilde{T})=o(\nu^*\pi_1^*v_{\mathbb{D}},\nu^*E)\smile o(\widetilde{\pi_2^*v_r},\widetilde{P}),$$

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where $\smile$ denotes the cup product. We have $o(\nu^*\pi_1^* v_{\mathbb{D}}, \nu^* E) = \nu^* \pi_1^* o(v_{\mathbb{D}}, T\mathbb{D})$. Since $\mathbb{B} \cap V = \mathbb{D} \times (\sigma \cap V)$ and $o(v_{\mathbb{D}}, T\mathbb{D})$ is a generator of $H^{2r-2}(\mathbb{D}, \partial\mathbb{D})$, we get

$$o(\nu^*\pi_1^* v_{\mathbb{D}}, \nu^* E) \frown [\nu^{-1}(\mathbb{B} \cap V), \nu^{-1}(\mathbb{S} \cap V)] = [\nu^{-1}(\sigma \cap V), \nu^{-1}(\partial\sigma \cap V)],$$

where $\frown$ denotes the cap product. Since the restriction of $o(\widetilde{\pi_2^* v_r}, \widetilde{P})$ to $\nu^{-1}\sigma$ is equal to $o(\widetilde{v_r}, \widetilde{P})$, comparing with (2.5), we obtain (2.6).

By Theorem 1.3, we have $\mathrm{Eu}(v_0, V; p) = \mathrm{Eu}(V, p) \cdot \mathrm{Ind}_{\mathrm{Sch}}(v_0, V; p)$. Also from definition, we have $\mathrm{Ind}_{\mathrm{Sch}}(v_0, V; p) = \mathrm{Ind}_{\mathrm{Sch}}(v_r, V; p) = \mathrm{Ind}_{\mathrm{Sch}}(v^{(r)}, V; p)$ and the theorem. $\quad\square$

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